

# EXPLICIT SOLVING OF THE SYSTEM OF NATURAL PDE'S OF MINIMAL SPACE-LIKE SURFACES IN MINKOWSKI SPACE-TIME

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**ABSTRACT.** A minimal space-like surface in Minkowski space-time is said to be of general type if it is free of degenerate points. The fact that minimal space-like surfaces of general type in Minkowski space-time admit canonical parameters of the first (second) type implies that any minimal space-like surface is determined uniquely up to a motion in space-time by the Gauss curvature and the normal curvature, satisfying a system of two PDE's (the system of natural PDE's). In fact this solves the problem of Lund-Regge for minimal space-like surfaces. Using canonical Weierstrass representations of minimal space-like surfaces of general type in Minkowski space-time we solve explicitly the system of natural PDE's, expressing any solution by means of two holomorphic functions in the Gauss plane. We find the relation between two pairs of holomorphic functions (i.e. the class of pairs of holomorphic functions) generating one and the same solution to the system of natural PDE's, i.e. generating one and the same minimal space-like surface in Minkowski space-time.

## 1. INTRODUCTION

We study space-like surfaces  $\mathcal{M}$  in Minkowski space-time  $\mathbb{R}_1^4$ , i.e. surfaces whose induced metric is of signature  $(2, 0)$ . The surface  $\mathcal{M}$  is minimal if its mean curvature vector field  $H$  is zero.

Let  $(\mathcal{M}, x(u, v))$  be a space-like surface in  $\mathbb{R}_1^4$ , parameterized by isothermal coordinates  $(u, v)$ . We denote by  $T_p(\mathcal{M})$  and  $N_p(\mathcal{M})$ , the tangential space and normal space at a point  $p \in \mathcal{M}$ , respectively. The second fundamental form on  $\mathcal{M}$  is denoted by  $\sigma$ .

A point  $p \in \mathcal{M}$  is said to be degenerate, if the set  $\{\sigma(X, Y); X \in T_p(\mathcal{M}), Y \in T_p(\mathcal{M})\}$ , is contained in one of the two light-like one-dimensional subspaces of  $N_p(\mathcal{M})$ .

A point  $p \in \mathcal{M}$  is degenerate if and only if the Gauss curvature  $K$  and the curvature of the normal connection  $\varkappa$  (the normal curvature) are zero at the point  $p$ .

We call a minimal space-like surface, free of degenerate points, a minimal space-like surface of general type.

The Gauss curvature  $K$  and the normal curvature  $\varkappa$  of a minimal space-like surface of general type, parameterized by special isothermal parameters, satisfy the following system of partial differential equations [1]:

$$(1.1) \quad \begin{aligned} (K^2 + \varkappa^2)^{\frac{1}{4}} \Delta \ln(K^2 + \varkappa^2)^{\frac{1}{4}} &= 2K, \\ (K^2 + \varkappa^2)^{\frac{1}{4}} \Delta \arctan \frac{\varkappa}{K} &= 2\varkappa. \end{aligned}$$

Conversely, any solution  $(K, \varkappa)$  to system (1.1) determines uniquely (up to a motion in  $\mathbb{R}_1^4$ ) a minimal space-like surface of general type with Gauss curvature  $K$  and normal curvature  $\varkappa$ .

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Further we call system (1.1) *the system of natural PDE's of minimal space-like surfaces in  $\mathbb{R}_1^4$*  and our aim is to solve explicitly this system.

All considerations in the paper are local.

In [4] we proved that any minimal space-like surface of general type admits locally canonical parameters of the first (second) type. The special isothermal parameters in [1] appear to be canonical parameters of the first type. These parameters are characterized by the second fundamental form in the following way:

$$(1.2) \quad \begin{aligned} \sigma(x_u, x_u) &\perp \sigma(x_u, x_v), \\ \sigma^2(x_u, x_u) - \sigma^2(x_u, x_v) &= 1. \end{aligned}$$

If we introduce canonical parameters on any minimal space-like surface of general type, then the Gauss curvature  $K$  and the normal curvature  $\varkappa$  satisfy the system of natural equations (1.1) and determine the surface up to a motion in  $\mathbb{R}_1^4$ . It is clear that the number of the invariants and the number of the PDE's can not be reduced further. Therefore this solves the problem of Lund-Regge [6] for minimal space-like surfaces in  $\mathbb{R}_1^4$ .

In this paper we prove the following theorems.

**Theorem 1.** *If the pair  $(K, \varkappa)$  is a solution to the system (1.1), then the curvatures  $K$  and  $\varkappa$  are given locally by the formulas*

$$(1.3) \quad \begin{aligned} K &= |\alpha| \operatorname{Re} \alpha; & \alpha &= \frac{-4g'_1 \bar{g}'_2}{(1 + g_1 \bar{g}_2)^2}, \\ \varkappa &= |\alpha| \operatorname{Im} \alpha; \end{aligned}$$

where  $(g_1, g_2)$  are holomorphic functions satisfying the conditions

$$(1.4) \quad g'_1 g'_2 \neq 0; \quad g_1 \bar{g}_2 \neq -1.$$

Conversely, any pair of holomorphic functions satisfying the conditions (1.4) generates by means of equalities (1.3) a solution to the system (1.1).

**Theorem 2.** *Two pairs of holomorphic functions  $(g_1, g_2)$  and  $(\hat{g}_1, \hat{g}_2)$  generate by means of (1.3) one and the same solution to the system (1.1), if and only if they are related by equalities of the type:*

$$(1.5) \quad \hat{g}_1 = \frac{ag_1 + b}{cg_1 + d}; \quad \hat{g}_2 = \frac{\bar{d}g_2 - \bar{c}}{-\bar{b}g_2 + \bar{a}},$$

where  $a, b, c, d \in \mathbb{C}$ ,  $ad - bc \neq 0$ .

## 2. EXPLICIT SOLVING OF THE SYSTEM OF NATURAL PDE'S OF MINIMAL SPACE-LIKE SURFACES OF GENERAL TYPE IN $R_1^4$

In [5] it is shown that using appropriate substitutions the above system (1.1) can be simplified. More precisely, putting  $K = e^{2X} \cos Y$ ;  $\varkappa = e^{2X} \sin Y$ , we get the following form of the natural equations:

$$(2.1) \quad \begin{aligned} \Delta X &= 2e^X \cos Y, \\ \Delta Y &= 2e^X \sin Y. \end{aligned}$$

It is easy to see that the system (2.1) of natural equations is equivalent to one complex partial differential equation:

$$(2.2) \quad \Delta(X + iY) = 2e^X (\cos Y + i \sin Y) = 2e^{X+iY}.$$

Putting in the last equation

$$(2.3) \quad \alpha = e^{X+iY},$$

we have:

$$(2.4) \quad \Delta \log \alpha = 2\alpha.$$

The last equation coincides formally with the natural partial differential equation (1.1) in [2] of space-like surfaces in  $\mathbb{R}_1^3$  with zero mean curvature:

$$(2.5) \quad \Delta \ln \nu - 2\nu = 0.$$

The difference between (2.4) and (2.5) is that  $\nu$  is a real positive function, while  $\alpha$  is a complex function. According to Theorem 6.2 in [2] any solution to (2.5) is given by the formula:

$$(2.6) \quad \nu(u, v) = \frac{\eta_u'^2(u, v) + \eta_v'^2(u, v)}{\eta^2(u, v)},$$

where  $\eta(u, v)$  is an arbitrary real harmonic function, satisfying the condition:  $\eta \neq 0$  and  $\eta_u'^2 + \eta_v'^2 \neq 0$ . By a direct substitution of (2.6) into (2.4) it is easy to check that this formula gives a solution also in the case, when  $\eta(u, v)$  is an arbitrary complex harmonic function. Taking into account (2.3) we use the following denotations:

$$(2.7) \quad e^X = |\alpha|; \quad e^X \cos Y = \operatorname{Re} \alpha; \quad e^X \sin Y = \operatorname{Im} \alpha.$$

Thus we found the following family of solutions to the system (1.1) of natural equations of minimal space-like surfaces of general type in  $\mathbb{R}_1^4$ :

$$(2.8) \quad \begin{aligned} K &= |\alpha| \operatorname{Re} \alpha; & \varkappa &= |\alpha| \operatorname{Im} \alpha; & \alpha &= \frac{\eta_u'^2 + \eta_v'^2}{\eta^2}, \end{aligned}$$

where  $\eta(u, v)$  is an arbitrary complex harmonic function, satisfying the conditions:  $\eta \neq 0$  and  $\eta_u'^2 + \eta_v'^2 \neq 0$ .

The following question arises naturally:

*Does the formula (2.8) give all the solutions to the system (1.1) of natural partial differential equations of minimal space-like surfaces in  $\mathbb{R}_1^4$ ?*

In [4] we proved the following statement:

**Theorem 2.1.** *Any minimal space-like surface  $\mathcal{M}$  of general type, parameterized by canonical coordinates of the first type, has the following Weierstrass representation:*

$$(2.9) \quad \Phi : \begin{aligned} \phi_1 &= i \frac{\cosh h_1}{\sqrt{h_1'^2 - h_2'^2}}, \\ \phi_2 &= \frac{\sinh h_1}{\sqrt{h_1'^2 - h_2'^2}}, \\ \phi_3 &= \frac{\cosh h_2}{\sqrt{h_1'^2 - h_2'^2}}, \\ \phi_4 &= \frac{\sinh h_2}{\sqrt{h_1'^2 - h_2'^2}}, \end{aligned}$$

where  $(h_1, h_2)$  are holomorphic functions satisfying the conditions:

$$(2.10) \quad h_1'^2 \neq h_2'^2; \quad \operatorname{Re} h_1 \neq 0 \text{ or } \operatorname{Im} h_2 \neq \frac{\pi}{2} + k\pi; \quad k \in \mathbb{Z}.$$

Conversely, if  $(h_1, h_2)$  is a pair of holomorphic functions satisfying the conditions (2.10), then formulas (2.9) generate a minimal space-like surface of general type, parameterized by canonical coordinates of the first type.

Introducing the functions:

$$w_1 = h_1 + h_2, \quad w_2 = h_1 - h_2; \quad g_1 = e^{w_1}; \quad g_2 = e^{w_2},$$

$$\theta = \operatorname{Re} h_1 + i \operatorname{Im} h_2,$$

then (2.9) implies that [4]:

$$(2.11) \quad K + i\kappa = \frac{-|\theta'_u|^2 + \theta'_v|^2 (\theta'^2_u + \theta'^2_v)}{|\cosh \theta|^2 \cosh^2 \theta},$$

$$(2.12) \quad K + i\kappa = \frac{-|w'_1 w'_2| w'_1 \bar{w}'_2}{\left| \cosh \frac{w_1 + \bar{w}_2}{2} \right|^2 \cosh^2 \frac{w_1 + \bar{w}_2}{2}},$$

$$(2.13) \quad K + i\kappa = \frac{-16|g'_1 g'_2| g'_1 \bar{g}'_2}{|1 + g_1 \bar{g}_2|^2 (1 + g_1 \bar{g}_2)^2}.$$

In terms of the functions  $w_1, w_2$  the canonical Weierstrass representation of minimal space-like surfaces of general type has the following form [4]:

**Theorem 2.2.** *Any minimal space-like surface  $\mathcal{M}$  of general type, parameterized by canonical coordinates of the first type, has the following Weierstrass representation:*

$$(2.14) \quad \Phi : \begin{aligned} \phi_1 &= \frac{i}{\sqrt{w'_1 w'_2}} \cosh \frac{w_1 + w_2}{2}, \\ \phi_2 &= \frac{1}{\sqrt{w'_1 w'_2}} \sinh \frac{w_1 + w_2}{2}, \\ \phi_3 &= \frac{1}{\sqrt{w'_1 w'_2}} \cosh \frac{w_1 - w_2}{2}, \\ \phi_4 &= \frac{1}{\sqrt{w'_1 w'_2}} \sinh \frac{w_1 - w_2}{2}. \end{aligned}$$

The functions  $(w_1, w_2)$  in this representation satisfy the following conditions

$$(2.15) \quad w'_1 w'_2 \neq 0; \quad w_1 + \bar{w}_2 \neq (2k + 1)\pi i; \quad k \in \mathbb{Z}.$$

Conversely, if  $(w_1, w_2)$  is a pair of holomorphic functions satisfying the conditions (2.15), then formulas (2.14) generate a minimal space-like surface of general type, parameterized by canonical coordinates of the first type.

In terms of the functions  $g_1, g_2$  the canonical Weierstrass representation of minimal space-like surfaces of general type has the following form [4]:

**Theorem 2.3.** *Any minimal space-like surface  $\mathcal{M}$  of general type, parameterized by canonical coordinates of the first type, has the following Weierstrass representation:*

$$(2.16) \quad \Phi : \begin{aligned} \phi_1 &= \frac{i}{2} \frac{g_1 g_2 + 1}{\sqrt{g'_1 g'_2}}, \\ \phi_2 &= \frac{1}{2} \frac{g_1 g_2 - 1}{\sqrt{g'_1 g'_2}}, \\ \phi_3 &= \frac{1}{2} \frac{g_1 + g_2}{\sqrt{g'_1 g'_2}}, \\ \phi_4 &= \frac{1}{2} \frac{g_1 - g_2}{\sqrt{g'_1 g'_2}}, \end{aligned}$$

where  $(g_1, g_2)$  is a pair of holomorphic functions satisfying the conditions:

$$(2.17) \quad g'_1 g'_2 \neq 0; \quad g_1 \bar{g}_2 \neq -1.$$

Conversely, if  $(g_1, g_2)$  is a pair of holomorphic functions satisfying the conditions (2.17), then formulas (2.16) generate a minimal space-like surface of general type, parameterized by canonical coordinates of the first type.

**2.1. Proof of Theorem 1.** Let  $(K, \varkappa)$  be a solution to the system (1.1). Then there exists a minimal space-like surface of general type, whose Gauss and normal curvatures are exactly the functions  $K$  and  $\varkappa$ , respectively. This surface has a canonical Weierstrass representation of the type (2.16). Applying formula (2.13), we obtain

$$(2.18) \quad \begin{aligned} K &= |\alpha| \operatorname{Re} \alpha; \\ \varkappa &= |\alpha| \operatorname{Im} \alpha; \end{aligned} \quad \alpha = \frac{-4g'_1 \bar{g}'_2}{(1 + g_1 \bar{g}_2)^2},$$

where  $(g_1, g_2)$  are a pair of holomorphic functions, satisfying the conditions  $g'_1 g'_2 \neq 0$  and  $g_1 \bar{g}_2 \neq -1$ .

Conversely, let the pair of holomorphic functions  $g_1, g_2$  satisfy the conditions (1.4). These two functions generate by means of (2.16) a minimal space-like surface of general type parameterized by canonical coordinates. The curvatures  $K$  and  $\varkappa$  of this surface from one side satisfy the system (1.1), from the other side they satisfy equality (2.13). Therefore the formulas (1.3) generate a solution to the system (1.1).  $\blacksquare$

The proof of Theorem 1 is based on Theorem 2.3. Next we give another forms of Theorem 1 based on Theorem 2.1 and Theorem 2.2.

Applying the canonical Weierstrass representation of the type (2.9) and formulas (2.11), we get the following statement.

**Theorem 1a.** *If the pair  $(K, \varkappa)$  is a solution to the system (1.1), then the curvatures  $K$  and  $\varkappa$  are given locally by the formulas*

$$(2.19) \quad \begin{aligned} K &= |\alpha| \operatorname{Re} \alpha; \\ \varkappa &= |\alpha| \operatorname{Im} \alpha; \end{aligned} \quad \alpha = \frac{-(\theta'_u{}^2 + \theta'_v{}^2)}{\cosh^2 \theta},$$

where  $\theta(u, v)$  is a complex harmonic function, satisfying the conditions  $\cosh \theta \neq 0$  and  $\theta'_u{}^2 + \theta'_v{}^2 \neq 0$ .

Conversely, any complex harmonic function  $\theta$ , satisfying the conditions  $\cosh \theta \neq 0$  and  $\theta'_u{}^2 + \theta'_v{}^2 \neq 0$  generates a solution to the system (1.1)

Further, applying the canonical Weierstrass representation of the type (2.14) and formulas (2.12), we have

**Theorem 1b.** *If the pair  $(K, \varkappa)$  is a solution to the system (1.1), then the curvatures  $K$  and  $\varkappa$  are given locally by the formulas*

$$(2.20) \quad \begin{aligned} K &= |\alpha| \operatorname{Re} \alpha; \\ \varkappa &= |\alpha| \operatorname{Im} \alpha; \end{aligned} \quad \alpha = \frac{-w'_1 \bar{w}'_2}{\cosh^2 \frac{w_1 + \bar{w}_2}{2}},$$

where  $(w_1, w_2)$  are holomorphic functions, satisfying the conditions  $\cosh^2 \frac{w_1 + \bar{w}_2}{2} \neq 0$  and  $w'_1 w'_2 \neq 0$ .

Conversely, any such a pair of holomorphic functions generates by means of the formulas (2.20) a solution to the system (1.1).

**2.2. Proof of Theorem 2.** Let us denote by  $\mathcal{M}$  and  $\hat{\mathcal{M}}$  the minimal space-like surfaces of general type given by (2.16). Then the condition that  $(g_1, g_2)$  and  $(\hat{g}_1, \hat{g}_2)$  generate one and the same solution to (1.1) means that  $\mathcal{M}$  and  $\hat{\mathcal{M}}$  have the same curvatures  $K$  and  $\varkappa$ . This is equivalent with the condition that  $\mathcal{M}$  and  $\hat{\mathcal{M}}$  are obtained one from the other by a rotation from the group  $\mathbf{SO}(3, 1, \mathbb{R})$  in  $\mathbb{R}_1^4$ .

In [?] we proved the following statement.

**Theorem 2.4.** *Let  $(\hat{\mathcal{M}}, \hat{x})$  and  $(\mathcal{M}, x)$  be two minimal space-like surfaces of general type, given by the canonical Weierstrass representation of the type (2.16). The following conditions are equivalent:*

- (1)  *$(\hat{\mathcal{M}}, \hat{x})$  and  $(\mathcal{M}, x)$  are related by a transformation in  $\mathbb{R}_1^4$  of the type:  
 $\hat{x}(t) = Ax(t) + b$ , where  $A \in \mathbf{SO}(3, 1, \mathbb{R})$  and  $b \in \mathbb{R}_1^4$ .*
- (2) *The functions in the Weierstrass representations of  $(\hat{\mathcal{M}}, \hat{x})$  and  $(\mathcal{M}, x)$  are related by the following equalities:*

$$(2.21) \quad \hat{g}_1 = \frac{ag_1 + b}{cg_1 + d}; \quad \hat{g}_2 = \frac{\bar{d}g_2 - \bar{c}}{-\bar{b}g_2 + \bar{a}},$$

where  $a, b, c, d \in \mathbb{C}$ ,  $ad - bc \neq 0$ .

Applying theorem 2.4, we obtain the assertion. ■

Finally we show how to derive (2.8) from formula (1.3).

Let us introduce the following pair  $(\xi_1, \xi_2)$  of holomorphic functions:

$$g_1 = \frac{1}{\xi_1}; \quad g_2 = \xi_2.$$

Then, by virtue of (1.3) we get the following formula for  $\alpha$ :

$$(2.22) \quad \alpha = \frac{4\xi'_1 \bar{\xi}'_2}{(\xi_1 + \bar{\xi}_2)^2},$$

where  $(\xi_1, \xi_2)$  is a pair of holomorphic functions satisfying the conditions  $\xi_1 + \bar{\xi}_2 \neq 0$  and  $\xi'_1 \xi'_2 \neq 0$ . Further we find a complex harmonic function  $\eta$ , which is related to the pair  $(\xi_1, \xi_2)$  in the same way as  $\theta$  is related to the pair  $(w_1, w_2)$ . Then  $\eta$  and  $(\xi_1, \xi_2)$  satisfy the same formulas as  $\theta$  and  $(w_1, w_2)$ .

Especially we have

$$(2.23) \quad \eta = \frac{\xi_1 + \bar{\xi}_2}{2}; \quad \eta'^2_u + \eta'^2_v = \xi'_1 \bar{\xi}'_2.$$

Conversely, if  $\eta$  is a complex harmonic function, we can obtain  $(\xi_1, \xi_2)$  in the same way as  $\theta$  gives  $(w_1, w_2)$ . Then (2.23) will be valid. Replacing  $(\xi_1, \xi_2)$  by means of  $\eta$  in (2.22) we find:

**Theorem 1c.** If the pair  $(K, \varkappa)$  is a solution to the system (1.1), then  $K$  and  $\varkappa$  are given by:

$$(2.24) \quad \begin{aligned} K &= |\alpha| \operatorname{Re} \alpha ; \\ \varkappa &= |\alpha| \operatorname{Im} \alpha ; \end{aligned} \quad \alpha = \frac{\eta_u'^2 + \eta_v'^2}{\eta^2},$$

where  $\eta(u, v)$  is a complex harmonic function, satisfying the conditions  $\eta \neq 0$  and  $\eta_u'^2 + \eta_v'^2 \neq 0$ .

Conversely, any such a function  $\eta$  gives by means of the formulas (2.24) a solution to the system (1.1).

Thus we obtained that (2.8) describes all solutions to the system (1.1).

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